General Topology and its Applications 6 (1976) 297–303

PRODUCTS WITH A COMPACT FACTOR

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Received 14 August 1974

In this paper we consider spaces $X \times Y$, where Y is a compact Hausdorff space. Most of this paper is devoted to giving new, simp ...ed proofs to some recent results concerning normality and map extension properties in ...e products. The theorems are of two types. First, we assume that the product $X \times Y$ is ...nal and deduce separation and covering properties for X, for example, that X must b... $w(Y)$-collectionwise normal. Second, we assume that X has some special separation p... ...erties (namely, $w(Y)$-collectionwise normality) and deduce some map extension p... ...perties for $X \times Y$. For example, if A and B are closed subsets of X and Y, respectively, ...en maps from $A \times B$ into the real line $\mathbb{R}$ can be extended to all of $X \times Y$ regardless of ...hether $X \times Y$ is normal or not. The proofs of all the theorems take advantage of the natur... l one-to-one correspondence between maps $f : X \times Y \to \mathbb{R}$ and maps $\tilde{f} : X \to C(Y)$.

AMS Subj. Class.: Primary 54C45, 54D15

normality in products	$C(X)$
collectionwise normal	extending maps

1. Introduction

All spaces here are Hausdorff topological spaces, and maps are continuous functions.

1.1. Definition. For a space Y let $C(Y)$ be the space of all bounded maps f from Y into the real line $\mathbb{R}$ with the sup norm topology, i.e., $\|f\| = \sup_{y \in Y} |f(y)|$. Let

$$B(f, \epsilon) = \{g \in C(Y) \mid d(f, g) = \sup_{y \in Y} |f(y) - g(y)| < \epsilon\}$$

be the open ϵ-ball about the point f in $C(Y)$.

In this paper we prove theorems about normality in product spaces $X \times Y$, where Y is a compact Hausdorff space. All the proofs about these product spaces make use of the rich structure of $C(Y)$, namely, that $C(Y)$ is a Banach space since Y is compact. In the applications, the weight of $C(Y)$ (see Definition 1.3(1)) is important, so in Section 2 we prove that for Y compact the weight of $C(Y)$ is equal to the weight of Y. These facts about $C(Y)$ are applied in our study of product spaces via the correspondence between $X \times Y$ and $C(Y)$ found in the following observation.

1.2. Observation. *If Y is a compact Hausdorff space, there is a natural one-to-one correspondence between maps $f : X \times Y \to \mathbf{R}$ and maps $\tilde{f} : X \to C(Y)$.* To a map $f : X \times Y \to \mathbf{R}$ we associate the map $\tilde{f} : X \to C(Y)$ defined by $\tilde{f}(x) = f \mid \{x\} \times Y$ for every $x \in X$. To a map $\tilde{f} : X \to C(Y)$ we associate the map $f : X \times Y \to \mathbf{R}$ defined by $f(x, y) = [\tilde{f}(x)](y)$. It is an easy exercise to check that the compactness of X makes maps correspond to maps [4, ch. XII, Theorem 3.1, p. 261].

I am indebted to K. Kunen and M.E. Rudin for many helpful discussions.

1.3. Definitions. (1) The *weight* of a space Y, denoted $w(Y)$, is the minimum cardinality of a basis for Y.

(2) A space is *λ-collectionwise normal* if λ is an infinite cardinal number and every discrete collection of closed sets containing λ or fewer closed sets can be mutually separated by disjoint open sets. Note that every normal space is ω_0-collectionwise normal.

(3) A space Y is *λ-paracompact* if λ is an infinite cardinal number and every open cover of Y by λ or fewer open sets has a locally finite open refinement.

2. Facts about $C(Y)$

The following two lemmas establish facts about the weight of $C(Y)$ which will be used in all of the succeeding sections.

2.1. Lemma. *If Y is a completely regular space, there is a collection of functions $\{f_\alpha\}_{\alpha < w(Y)}$ in $C(Y)$ such that for every $\alpha < w(Y)$, $\|f_\alpha\| = 1$ and for $\alpha < \beta < w(Y)$, $d(f_\alpha, f_\beta) \geq 1$.*

Proof (simplified by W. Rudin). We choose the f_α's inductively as follows: Suppose $\beta < w(Y)$ and f_α has been chosen for each $\alpha < \beta$. Let $U_\alpha = f_\alpha^{-1}((0, 1])$. Since $\beta < w(Y)$, $\{U_\alpha\}_{\alpha<\beta}$ is not a basis for Y. Therefore, there is a point $y_\beta \in Y$ with a neighbourhood W_β such that no U_α containing y_β lies in W_β. Define a map $f_\beta : Y \to [-1, 1]$ so that $f_\beta(y_\beta) = 1$ and $f_\beta(Y - W_\beta) = -1$.

Now suppose $\alpha < \beta < w(Y)$. If $f_\alpha(y_\beta) \leqslant 0$, then

$$f_\alpha(y_\alpha) - f_\beta(y_\alpha) = 1 - f_\beta(y_\alpha) \geqslant 1 .$$

If $f_\alpha(y_\beta) > 0$, then $y_\beta \in U_\alpha$, so there is a point $y \in U_\alpha - W_\beta$. Thus

$$f_\alpha(y) - f_\beta(y) = f_\alpha(y) + 1 > 1 .$$

So $d(f_\alpha, f_\beta) \geqslant 1$, and the lemma is proved. □

2.2. Lemma. *If Y is an infinite, compact Hausdorff space, then* $w(C(Y)) = w(Y)$.

Proof. The sup norm topology on $C(Y)$ is the same as the compact-open topology on $C(Y)$ which has as a subbase the sets $\{(\bar{U}_\alpha, V_i)\}_{\alpha<w(Y), i\in\omega_0}$, where U_α is a basic open set for Y, V_i is a basic open set in $\mathbb{R}$, and $(\bar{U}_\alpha, V_i) = \{f : Y \to \mathbb{R} \mid f(\bar{U}_\alpha) \subset V_i\}$. Thus $w(C(Y)) \leqslant w(Y)$. Lemma 2.1 provides the reverse inequality, so the lemma is proved. □

3. Implications of the normality of a product with a compact factor

When the product of two spaces is normal, it is possible to deduce facts about the covering and separation properties of the factors. A classical example of this type of theorem is Tamano's theorem that if the product of a completely regular T_1 space X and its Stone–Čech compactification βX is normal, then X is paracompact [13, Theorem 2].

In the following two theorems we assume, for a compact space Y, that $X \times Y$ is normal, and deduce, first, that X is $w(Y)$-collectionwise normal and, second, that X has a covering property related to, but weaker than, $w(Y)$-paracompactness. The proofs below are simple because we make use of the metrizability and weight of $C(Y)$. The next two theorems are recent results of M.E. Rudin and have more difficult proofs in [8, Theorem 2 and Theorem 3, respectively].

3.1. Theorem. *If Y is a compact Hausdorff space and $X \times Y$ is normal, then X is $w(Y)$-collectionwise normal.*

Proof. Let $\{A_\alpha\}_{\alpha < w(Y)}$ be a discrete collection of closed sets in X. Let $A = \bigcup_{\alpha < w(Y)} A_\alpha$. Choose $\{f_\alpha\}_{\alpha < w(Y)}$ as in Lemma 2.1 and define the map $g : A \to C(Y)$ by $g(x) = f_\alpha$ for $x \in A_\alpha$. Using 1.2, we get a map $\tilde{g} : A \times Y \to \mathbf{R}$. By the Tietze extension theorem, there is a map $\tilde{G} : X \times Y \to \mathbf{R}$ extending $\tilde{g}$. By 1.2, we obtain a map $G : X \to C(Y)$ extending g. But then $\{G^{-1}(B(f_\alpha, \frac{1}{2}))\}_{\alpha < w(Y)}$ is a collection of disjoint open sets in X which separate the A_α's, proving the theorem. □

3.2. Scholium. *If Y is a compact Hausdorff space and X is not $w(Y)$-collectionwise normal, then there is a closed subset A of X and a map $f : A \times Y \to \mathbf{R}$ which cannot be extended to a map $F : X \times Y \to \mathbf{R}$.*

The Scholium is a direct consequence of the proof of Theorem 3.1.

The next theorem is M.E. Rudin's characterization of normality in a product with a compact factor.

3.3. Theorem. *If Y is a compact Hausdorff space, then $X \times Y$ is normal if and only if for every pair of disjoint closed sets H and K in $X \times Y$ there is a locally finite closed cover $\{C_\alpha\}_{\alpha < w(Y)}$ of X and open sets $\{U_\alpha, V_\alpha\}_{\alpha < w(Y)}$ in Y such that for every $\alpha < w(Y)$:*

(i) $U_\alpha \cap V_\alpha = \emptyset$;
(ii) $H \cap (C_\alpha \times Y) \subset C_\alpha \times U_\alpha$;
(iii) $K \cap (C_\alpha \times Y) \subset C_\alpha \times V_\alpha$.

Proof. ($\Rightarrow$) Let $f : X \times Y \to [0, 1]$ be a map such that $H \subset f^{-1}(0)$ and $K \subset f^{-1}(1)$. This map induces a map $\tilde{f} : X \to C(Y)$. By Lemma 2.2 and the fact that $C(Y)$ is a metric space, we can let $\{W_\alpha\}_{\alpha < w(Y)}$ be a locally finite closed refinement of the cover of $\tilde{f}(X)$ by balls of radius $\frac{1}{4}$ in $C(Y)$. For each α, pick a point $g_\alpha \in W_\alpha$ and let $U_\alpha = g_\alpha^{-1}([0, \frac{1}{2}))$, $V_\alpha = g_\alpha^{-1}((\frac{1}{2}, 1])$, and let $C_\alpha = \tilde{f}^{-1}(W_\alpha)$. These are the sets required in the theorem.

($\Leftarrow$) This direction is not difficult and appears in [8].

4. Extension theorems

The Tietze extension theorem tells us that for every closed subset A of a normal space N and every map f from A into $\mathbf{R}$ there is a map F from N

into **R** which extends f. In the next theorem we consider product spaces $X \times Y$, where Y is a compact Hausdorff space and X is a $w(Y)$-collectionwise normal space. We show that maps into **R** defined on closed subsets of $X \times Y$ which are themselves products can be extended to all of $X \times Y$ regardless of whether $X \times Y$ is normal or not. In the proof of this theorem the Observation 1.2 is again used to allow us to take advantage of $C(Y)$. Here, however, we make use of $C(Y)$ not only as a metric space with a certain weight, but also as a Banach space endowed with certain extension properties of its own (see Theorem D below).

4.1. Theorem. *If Y is a compact Hausdorff space, X is a $w(Y)$-collectionwise normal space, and A and B are closed subsets of X and Y respectively, then any map $f : A \times B \to \mathbf{R}$ can be extended to a map $F : X \times Y \to \mathbf{R}$.*

Proof. The proof has two steps. First, we extend f to a map $f' : A \times Y \to \mathbf{R}$. Next we extend f' to the desired map F.

To obtain f', we apply the following theorem [11, Theorem 2] (whose proof, incidentally, makes use of 1.2). Note that Theorem S is a special case of Theorem 4.1.

Theorem S. *Let B be a closed subset of a compact Hausdorff space Y and let A be any space. Then any map $f : A \times B \to \mathbf{R}$ can be extended to a map $f' : A \times Y \to \mathbf{R}$.*

We will now assume that the map $f' : A \times Y \to \mathbf{R}$ has been obtained and proceed to extend it to the map $F : X \times Y \to \mathbf{R}$ demanded in Theorem 4.1. Alo and Sennott [1, Corollary 3.6] showed that f' can be so extended. In order to use 1.2 again, here we prove that f' can be extended by making use of the following theorem about Banach spaces essentially due to Dowker [3, Lemma 2, Theorem 2]. (Dowker actually states his Lemma 2 for a Hilbert space E and without the $w(E)$ prefix, but his proof suffices to yield the following theorem.)

Theorem D. *If E is a Banach space, X is a $w(E)$-collectionwise normal space, and A is a closed subset of X, then every map $g : A \to E$ can be extended to a map $G : X \to E$.*

We use Dowker's theorem to extend f' as follows. Use 1.2 to obtain a map $\tilde{f} : A \to C(Y)$. By Lemma 2.2, $w(C(Y)) = w(Y)$. Hence, since $C(Y)$ is a Banach space and X is $w(Y)$-collectionwise normal, Theorem D can be applied to obtain a map $\tilde{F} : X \to C(Y)$. One more application of 1.2 gives us the desired map $F : X \times Y \to \mathbf{R}$. □

5. Borsuk's Homotopy Extension Theorem

One of the hypotheses of Borsuk's Homotopy Extension Theorem [9, D2, p. 57] is that a product space $X \times I$ (where I is $[0, 1]$) is normal. This hypothesis led Dowker to investigate the normality of spaces $X \times I$ [2]. In [2] Dowker demonstrated the close relationship between normality in $X \times I$ and countable paracompactness in X. Using Dowker's theorem, Rudin [7] produced an example of a normal space X whose product with an interval is not normal. Thus the statement that $X \times I$ is normal is definitely stronger than the statement that X is normal.

Recently, Starbird [12] and (independently) Morita [5, Theorem 7] have proved some versions of the Borsuk Homotopy Extension Theorem which remain true even when the traditional hypothesis that $X \times I$ is normal is replaced by the weaker hypothesis that X is normal.

In this section we give a simplified proof of one such version of Borsuk's theorem [12, Theorem 1]. The proof makes use of the following special case of Theorem 4.1. The following theorem was first proved by Alo and Sennott [1, Corollary 3.5].

5.1. Theorem. *If Y is a compact metric space, X is a normal space, and A is a closed subset of X, then any map $f : A \times Y \to \mathbf{R}$ can be extended to a map $F : X \times Y \to \mathbf{R}$.*

This theorem is a special case of Theorem 4.1 since every normal space is w_0-collectionwise normal.

Note that Theorem 5.2 below would be a direct consequence of the Tietze extension theorem if we assumed that $X \times I$ were normal rather than just that X is normal. Theorem 5.2 is a strengthening of Theorem 5.1 and is the key to all the versions of Borsuk's theorem which appear in [12].

5.2. Theorem. *If A is a closed subset of a normal space X and $f : A \times I \cup X \times \{0\} \to \mathbf{R}$ is a map, then there exists a map $F : X \times I \to \mathbf{R}$ extending f.*

Proof. By Theorem 5.1 there is a map $\widetilde{F} : X \times I \to \mathbf{R}$ extending $f | A \times I$. The only difficulty is that $\widetilde{F}$ may not agree with f on $X \times \{0\}$. Therefore we need to modify $\widetilde{F}$. Let $H : X \times I \to \mathbf{R}$ be the straight line homotopy between $f | X \times \{0\}$ and $\widetilde{F} | X \times \{0\}$. For every point $x \in X$, we will define our final map $F | \{x\} \times I$ as follows. Let $t = \min\{|\widetilde{F}(x, 0)$

$-f(x,0)|, \frac{1}{2}\}$. In the interval $(x, 0)$ to (x, t) squeeze in the map $H | \{x\} \times I$. In the interval (x, t) to $(x, 1)$ squeeze in the map $\widetilde{F} | \{x\} \times I$. Notice that for points $x \in X$ where $\widetilde{F}(x, 0) = f(x, 0)$ the map $F | \{x\} \times I$ equals $\widetilde{F} | \{x\} \times I$. Therefore, since $\widetilde{F} | A \times I = f | A \times I$, the map $F : X \times I \to \mathbf{R}$ obtained above is the desired extension of f. □

In [6, Theorem 3.7], Morita and Hoshina show that in fact Theorem 5.2 can be improved by replacing I by any compact metric space M and replacing $\{0\}$ by any closed subset of M. Their theorem also implies the following variation of Theorem 4.1:

If Y is a compact Hausdorff space, X is a $w(Y)$-collectionwise normal space, A and B are closed subsets of X and Y respectively, then any map $f : (A \times Y) \cup (X \times B) \to \mathbf{R}$ can be extended to a map $F : X \times Y \to \mathbf{R}$.

References

[1] R.A. Alo and L.I. Sennott, Collectionwise normality and the extension of functions on product spaces, Fund. Math. 76 (1972) 231–243.
[2] C.H. Dowker, On countably paracompact spaces, Can. J. Math. 3 (1951) 219–224.
[3] C.H. Dowker, On a theorem of Hanner, Ark. Mat. 2 (1952) 307–313.
[4] J. Dugundji, Topology (Allyn and Bacon, Boston, Mass., 1966).
[5] K. Morita, On generalizations of Borsuk's homotopy extension theorem, Fund. Math. 88 (1975) 1–6.
[6] K. Morita and T. Hoshina, C-embedding and the homotopy extension property, Gen. Topology Appl. 5 (1975) 69–81.
[7] M.E. Rudin, A normal space X for which $X \times I$ is not normal, Fund. Math. 73 (1971) 179–186.
[8] M.E. Rudin, The normality of products with one compact factor, Gen. Topology Appl. 5 (1975) 45–59.
[9] E.H. Spanier, Algebraic Topology (McGraw-Hill, New York, 1966).
[10] M. Starbird, The normality of products with a compact or a metric factor, Ph.D. Thesis, Univ. of Wisconsin (1974).
[11] M. Starbird, Extending maps from products, in: N.M. Stavrakas and K.R. Allen, eds., Studies in Topology (Academic Press, New York, 1975) 559–564.
[12] M. Starbird, The Borsuk homotopy extension theorem without the binormality condition, Fund. Math. 87 (1975) 207–211.
[13] H. Tamano, On paracompactness, Pacific J. Math. 10 (1960) 1043–1047.